Fluctuations in the mean-field Ising model

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Structure

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The Ising Model

Consider,

$$\mathbb{P}^{N}_{\beta,B}(\sigma=\tau) := \frac{1}{Z_N(\beta,B)}\exp\left(\frac{\beta}{2\bar{d}_N}\tau' A(G_N)\tau + B\sum_{i=1}^{N}\tau_i\right) \quad (1)$$

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2. $\bar{d}_N$ – Average degree of G_N, i.e., $\bar{d}_N = (1/N)\sum_i d_i$.
3. (β, B) – parameters in $[0, \infty) \times (-\infty, \infty)$.
4. We observe one copy of (spins) σ.

The Ising Model (continued)

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4. $B = 0$ implies the absence of external influence.
5. $(A(G_N))_{ij} > 0$ implies sites i and j are inclined to align in the same direction.

Assumptions

(A1) (Bounded row sums):

$$\limsup_{N\to\infty} \max_{1\le i\le N} \frac{d_i}{\overline{d}_N} < \infty.$$

(A2) (Well connectedness):

$$\limsup_{N\to\infty} \lambda_2\left(\frac{A(G_N)}{\overline{d}_N}\right) < 1.$$

Commonly studied examples

$A(G_N)$ is the adjacency matrix of:

- Lattice graph in d dimensions: Original paper by Ising (1925) considered the one dimensional lattice. It was proposed by Lenz to model sharp phase transition of ferromagnetic properties (at Curie temperature).
- Complete graphs: Curie-Weiss model.
- Random graphs: Erdős Rényi, random regular, etc.

Applications

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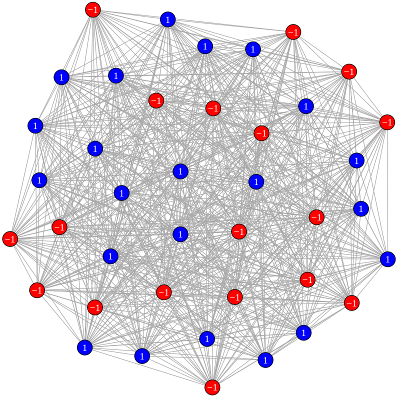
2. In social networks, to study trends in opinions (voting choices), where G_N could be determined by "friendships" within the network.
3. Modeling ferromagnetic properties (i.e., **sharp** change in magnetic properties of magnetic materials when heated beyond a certain (**Curie**) temperature).

$N = 40$, $p = 0.9$, $\beta = 0.5$, $B = 0$, seed=7.11



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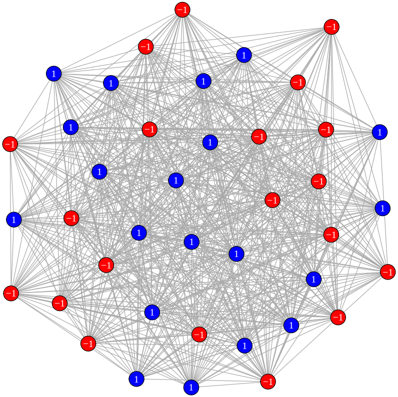

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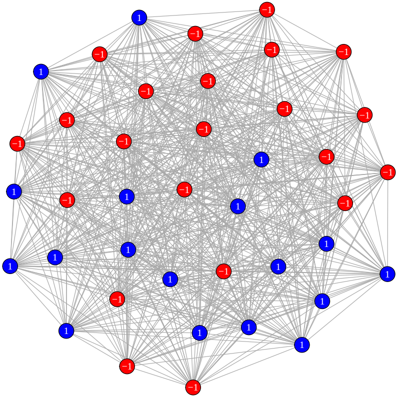

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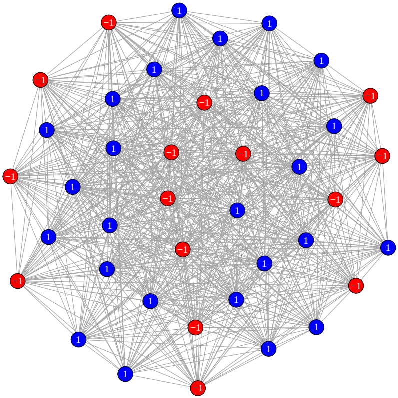

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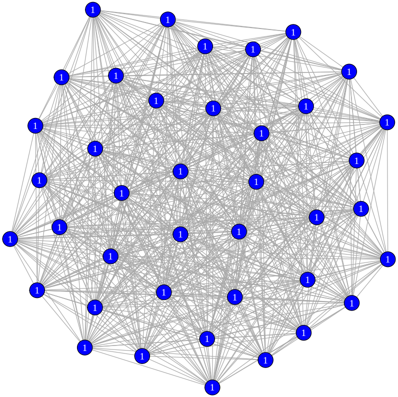

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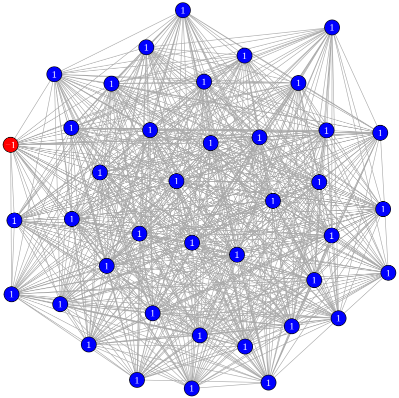

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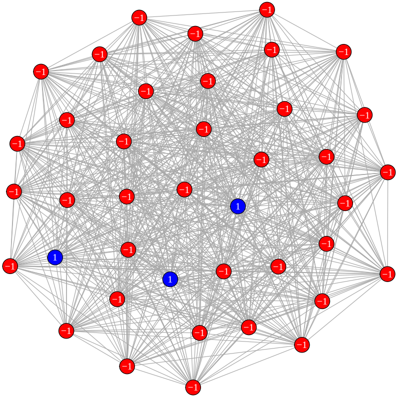

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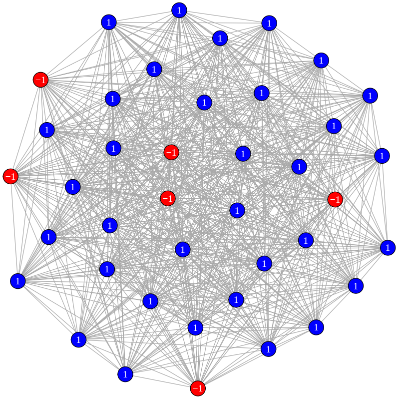

Applications continued...

- Physics: Ising [1925], Fytas et al. [2019].
- Social Sciences: Marsman et al. [2018], Schelling [1971].
- Neuroscience: Yuste [2015], Fraiman et al. [2009].
- Image analysis: Besag [1986], Sun et al. [2012].
- Machine Learning: Hopfield [1982], Zhang et al. [2015].
- Earth Sciences: Siegel et al. [2018], Raymond et al. [2016].
- Theoretical (goodness of fit): Martín del Campo et al. [2017], Daskalakis et al. [2018].

Statistical motivation

Motivating Theorem, Barndorff [2014]

Suppose $\alpha \in \mathbb{R}$, $\phi : \mathbb{R}^n \mapsto \mathbb{R}$ and $\mathcal{P}_\mu(\mathbb{R}^n)$: prob. measures on $\mathbb{R}^n$, $\ll \mu$. Then,

$$\max_{\mathcal{P}_\mu(\mathbb{R}^n)} -\int p(x) \log p(x)\, d\mu(x) \text{ subject to } \mathbb{E}_p \phi(\cdot) = \alpha$$

is attained at uniquely at $p^*_\theta(x) \propto \exp(\theta\phi(x))$ where $\mathbb{E}_{p^*_\theta}\phi(\cdot) = \alpha$.

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Put constraints on pairwise weighted correlations (simple interaction): $\sum_{i,j} w_{ij}\sigma_i\sigma_j$ (from the data). Then $p^*(\sigma) \propto \exp(\beta \sum_{i,j} w_{ij}\sigma_i\sigma_j)$ ((1) with $B = 0$).

Structure

Literature Review

Recall, for $\tau \in \{-1, 1\}^N$,

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- (Main motivation): Basak and Mukherjee [2017] show that,

$$\frac{1}{N}\log Z_N(\beta,B) \tag{2}$$

has an "universal limit" (free of the underlying graph sequence) as long as G_N is "approximately" regular and $\bar{d}_N$ diverges to ∞ with N.

Regular graphs

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- Note that $(A_N)_{ij} = 1/\overline{d}_N$ if i and j are neighbors in G_N.
- $\sum_j (A_N)_{ij} = d_i/\overline{d}_N = 1$ for all $i \in [N]$ (equivalently $\mathbf{1}'A_N = \mathbf{1}'$) : **(P1)**.
- By Perron-Frobenius, $\lambda_1(A_N) = 1$: **(P2)**.

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- Our methods also work under random graphs, such as Erdős Rényi above.
- Lower bounds on the "extent of regularity".

Variational Approach to "universality"

- Given $\mathbb{Q}_N(\mathbf{x})$ be a discrete measure supported on $\{-1,1\}^N$. Then $D(\mathbb{Q}_N \| \mathbb{P}_N)$ equals,

$$\sum_{\mathbf{x}\in\{-1,1\}^N} \mathbb{Q}_N(\mathbf{x}) \log \frac{\mathbb{Q}_N(\mathbf{x})}{\mathbb{P}_N(\mathbf{x})}$$

$$= \sum_{\mathbf{x}\in\{-1,1\}^N} \mathbb{Q}_N(\mathbf{x}) \log \mathbb{Q}_N(\mathbf{x}) - \frac{\beta}{2} \sum_{\mathbf{x}\in\{-1,1\}^N} \mathbb{Q}_N(\mathbf{x}) \mathbf{x}' A_N \mathbf{x}$$

$$- B \sum_{\mathbf{x}\in\{-1,1\}^N} \mathbb{Q}_N(\mathbf{x}) \mathbf{1}'\mathbf{x} + \log Z_N(\beta, B)$$

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- As $D(\cdot\|\cdot) \geq 0$,

$$\log Z_N(\beta, B) \geq \mathbb{E}_{\mathbb{Q}_N}\left\{\frac{\beta}{2}\mathbf{x}' A_N \mathbf{x} + B\mathbf{1}'\mathbf{x} - \log \mathbb{Q}_N(\mathbf{x})\right\}.$$

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- Choose $\mathbb{Q}_N(\mathbf{x})$ as the measure induced by independent ± 1 rv's with mean vector $\mathbf{m} := (m_1, m_2, \ldots, m_N)$.

$$\log Z_N(\beta, B) \geq \frac{\beta}{2}\mathbf{m}' A_N \mathbf{m} + \sum_{i=1}^{N}(Bm_i - I(m_i))$$

  for any $\mathbf{m}$, where $I(x) := \frac{1+x}{2}\log\frac{1+x}{2} + \frac{1-x}{2}\log\frac{1-x}{2}$ is the binary entropy function.

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$$\sup_{\mathbf{m}} \frac{\beta}{2}\mathbf{m}' A_N \mathbf{m} + \sum_{i=1}^{N} (Bm_i - I(m_i))$$

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- Now restrict to "regular graphs". By **(P2)**,

$$\sup_{\mathbf{m}} \left\{ \frac{\beta}{2}\mathbf{m}' A_N \mathbf{m} + \sum_{i=1}^{N} (Bm_i - I(m_i)) \right\}$$

$$\leq \sup_{\mathbf{m}} \sum_{i=1}^{N} \left\{ \frac{\beta}{2} m_i^2 + Bm_i - I(m_i) \right\}$$

which is decoupled.

Variational Approach towards “universality”

- The supremum on RHS is attained at $\mathbf{m} = t\mathbf{1}'$ where $t > 0$ satisfies $t = \tanh(\beta t + B)$. Therefore,

$$\sup_{\mathbf{m}} \left\{ \frac{\beta}{2}\mathbf{m}' A_N \mathbf{m} + \sum_{i=1}^{N} (Bm_i - I(m_i)) \right\}$$

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- By **(P1)**, $\mathbf{m}' A_N \mathbf{m} = Nt^2$ at $\mathbf{m} = t\mathbf{1}'$. So,

$$\sup_{\mathbf{m}} \left\{ \frac{\beta}{2}\mathbf{m}' A_N \mathbf{m} + \sum_{i=1}^{N} (Bm_i - I(m_i)) \right\} = N \left\{ \frac{\beta t^2}{2} + Bt - I(t) \right\}.$$

Variational Approach towards "universality"

- Combining above observations,

$$\frac{\log Z_N(\beta, B)}{N} \geq \frac{\beta t^2}{2} + Bt - I(t) \tag{3}$$

for regular graphs. The RHS above is often called "mean-field prediction".

- Note that the right hand side no longer depends on A_N ("universality") for the whole spectrum of (β, B).

Actual Theorem

Theorem in Basak and Mukherjee [2017] (universality)

Under model (1) (and **(A1)**), if G_N is a sequence of "approximately" regular graphs, with $\bar{d}_N \xrightarrow{N\to\infty} \infty$, then:

$$\frac{1}{N}\log Z_N(\beta, B) \xrightarrow{N\to\infty} \sup_x \left(\frac{\beta}{2}x^2 + Bx - I(x)\right).$$

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- (Lower bound) There are bounded degree graph sequences for which the "mean-field prediction" does not hold.

Statistic of interest

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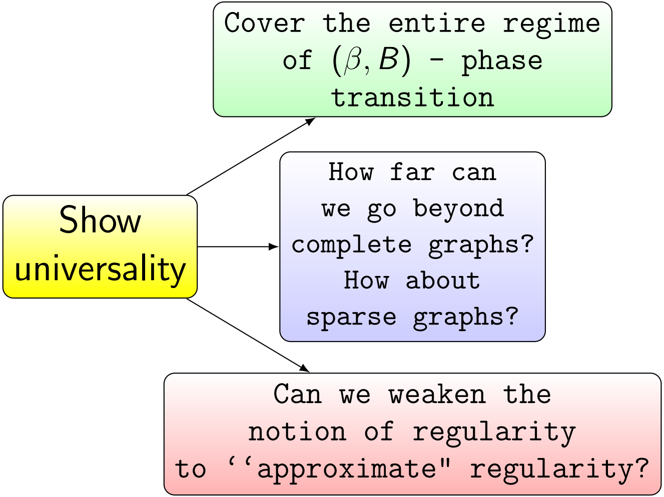
Is the behaviour of T_N universal?

Our goal


Show universality
Cover the entire regime of (β, B) - phase transition
How far can we go beyond complete graphs? How about sparse graphs?
Can we weaken the notion of regularity to ''approximate'' regularity?

Weak Limits

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2. If $\beta > 1, B = 0$ and G_N, then

$$T_N \xrightarrow{\mathbb{P}^{\mathbb{N}}_{\beta,\mathbb{B}}} \begin{cases} t & \text{w.p. } 0.5 \\ -t & \text{w.p. } 0.5 \end{cases}.$$

Here $t \geq 0$ is the maximizer of the RHS in (3). For non-uniqueness, **(A2)** is required.

A natural question

Does universality extend beyond $Z_N(\beta, B)$ and the weak limits for T_N?

$$N^{?}(T_N - t) \xrightarrow{w} \text{ universal limit distributions}$$

(perhaps after conditioning).

Our results

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(i). (Uniqueness) If $\beta < 1, B = 0$ or $B > 0$,

$$\sqrt{N}(T_N - t) \xrightarrow{d} \mathcal{N}\left(0, \frac{1-t^2}{1-\beta(1-t^2)}\right).$$

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(iii). (Critical) if $B = 0, \beta = 1$ (and **(A2)** holds),
$$N^{1/4} T_N \xrightarrow{d} W(\text{ density } \propto \exp(-x^4/12)).$$

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Eg. Consider $\bar{d}_N = \lambda\sqrt{N}$ and G_N as a disjoint union of $N/\bar{d}_N$ complete graphs, each of size $\bar{d}_N$, then we prove (for $B > 0$):

$$\sqrt{N}(T_N - t) \xrightarrow{w} \mathcal{N}\left(\frac{\mu(\beta, B)}{\lambda}, \frac{1-t^2}{1-\beta(1-t^2)}\right).$$

$\mu(\beta, B) \neq 0$ and free of λ.

Some background

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- The same problem of fluctuations was studied in Ellis and Newman [1978] when G_N is a complete graph. Our result strengthens it considerably.
- In Löwe et al. [2018] and Berthet et al. [2016], the authors study fluctuations in "block-spin Ising models".

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where $m_i = \sum_j a_{ij}\sigma_j$ (average response from neighbours).

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Look at $S_n := \frac{1}{N}\sum_i(\sigma_i - \tanh(\beta m_i + B))$. For bounded $\bar{d}_N$, S_N was analysed in Comets and Janžura [1998].

Conditionally centered fluctuation

Theorem 2 (universality)

Under model (1) (and **(A1)**), if G_N is a sequence of "approximately" regular graphs, with $\bar{d}_N \xrightarrow{N\to\infty} \infty$, then for any (β, B):

$$\sqrt{N}S_N = \frac{1}{\sqrt{N}}\sum_i (\sigma_i - \tanh(\beta m_i + B)) \xrightarrow{d} \mathcal{N}(0, \tau^2).$$

Here $\tau^2 = (1-t^2)(1-\beta(1-t^2))$. For the non-uniqueness regime, **(A2)** is required.

Conditionally centered fluctuation

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When $\beta = 1, B = 0$ however, $\tau = 0$. We expect in that case, $N^{3/4} S_n = O_p(1)$.

"Ising" on the cake?

For Theorem 1, we actually get Berry-Esseen equivalents for the weak-convergence, e.g., if $\beta < 1, B = 0$ or $B > 0$, then:

$$\sup_x \left|\mathbb{P}(\sqrt{N}(T_N - t) \leq x) - \mathbb{P}(\sigma Z \leq x)\right| \lesssim \sqrt{N}/\bar{d}_N.$$

Here σ^2 is the asymptotic variance from Theorem 1. Same rates hold for the other regimes of (β, B).

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1. If $\bar{d}_N \approx N$ (dense graphs), then RHS$\lesssim N^{-1/2}$.
2. If $\bar{d}_N \approx \sqrt{N}$, then RHS$\lesssim O(1)$ (we know that the result is not true in this regime).

More implications

1. In terms of parameter estimation (β, B), we improve the existing results from Ghosal and Mukherjee [2018] and Bhattacharya and Mukherjee [2018] (e.g., Extending impossibility regimes).

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1. In terms of parameter estimation (β, B), we improve the existing results from Ghosal and Mukherjee [2018] and Bhattacharya and Mukherjee [2018] (e.g., Extending impossibility regimes).
2. We prove that, marginally,

$$\sqrt{N}(\tilde{\beta} - \beta) \xrightarrow{d} \mathcal{N}\left(0, \frac{1 - \beta(1 - t^2)}{1 - t^2}\right)$$

$$\sqrt{N}(\tilde{B} - B) \xrightarrow{d} \mathcal{N}\left(0, \frac{1 - \beta(1 - t^2)}{1 - t^2}\right)$$

for certain regimes. $\tilde{\beta}$ represents pseudo-likelihood estimator (for B known), see Besag [1986].

Implications continued

Our results extend to graphs which are "approximately" regular (with counterexamples showing tightness for the extent of regularity). In addition to deterministic graphs, we also cover:

1. Random $d-$regular graphs.
2. $G_N :=$ Erdős-Rényi graph with parameters (N, p_N) if $p_N \gg (\log N)N^{-1}$.
3. $G_N :=$ stochastic block model with balanced, possibly growing, block sizes.
4. $W-$random graphons (where $W(\cdot)$ is regular).


The End

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